The rank of the endomorphism monoid of uniformly nested partition.

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April 23, 2010

Introduction

Let S be a semigroup. We say that X *generates* S if every element of S can be written as a product of elements in X. The least possible cardinality $\mathrm{rk}\,(S)$ of a generating set of S is called the *rank* of S.

There is a variety of works where the ranks of transformation monoids of different structures are explicitly computed. In particular, in [1] the rank of endomorphism monoid of uniform partition was shown to be 4. In this paper we consider the endomorphism monoid of uniformly nested partition and show that its rank is $2k$, where k is the depth of nesting.

1 Nested partitions

First we fix definition related to the notion of tree. As usually *a (rooted) tree* T is a simply connected graph with a fixed vertex $\perp$ called root. We will indicate by $t \in T$ that t is a vertex of T. For t_1, $t_2 \in T$ we define $d(t_1, t_2)$ to be the number of edges at the unique path from t_1 to t_2. Note that d gives a distance function on the set of vertices of T. We define a level of $t \in T$ as a distance from t to root. Denote by T_k the set of all vertices in T of level k. A vertex $s \in T_{k+1}$ is called *a child* of $t \in T_k$ if there is an edge between s and t.

A *nested partition* of a set X is a collection of subsets $\{\, P_t \,|\, t \in T\}$ of X parametrized by the vertices of a tree T such that

a) $P_\perp = X$;

b) for any non-leaf $t \in T$:

$$P_t = \coprod_{s \text{ child of } t} P_s.$$

*The work is supported by the FCT Grant SFRH/BPD/31788/2006. The financial support by CMUC and FCT gratefully acknowledged.

We say that a map $f\colon X \to Y$ *respects* nested partitions $\{\, Q_s \subset X \mid s \in S\}$ and $\{\, P_t \subset Y \mid t \in T\}$ if for every $s \in S$ exists (necessarily unique) $t \in T$ of the same level as s such that $f\,(Q_s) \subset T_t$.

For every nested partition $\{\, P_t \mid t \in T\}$ of X we define sets X_k by

$$X_k = \left(\coprod_{\substack{t \in T_k \\ t \text{ non-leaf}}} \{\text{childs of } t\} \right) \amalg \left(\coprod_{\substack{t \in T_k \\ t \text{ leaf}}} P_t \right)$$

and maps $\rho_k\colon X_{k+1} \to X_k$ by

$$\begin{aligned} \rho_k(s) &= t, && \text{if } s \in T_{k+2} \text{ is a child of } t \in T_{k+1} \\ \rho_k(x) &= t, && \text{if } t \in T_{k+1} \text{ is a leaf and } x \in P_t. \end{aligned}$$

Note that if T is a tree of depth k, then $X_l = \varnothing$ for $l \geq k+1$.

If $f\colon X \to Y$ respects partitions $\{\, Q_s \subset X \mid s \in S\}$, $\{\, P_t \subset Y \mid t \in T\}$ define $f_k\colon X_k \to Y_k$ by

$$\begin{aligned} f_k(s) &= t && \text{if } s \in T_{k+1} \text{ and } f\,(Q_s) \subset P_t \\ f_k(x) &= f(y) && \text{if } s \in S_k \text{ is a leaf and } x \in P_s. \end{aligned}$$

Proposition 1. *The diagrams*

$$\begin{tikzcd} X_{k+1} \arrow[r, "\rho_k"] \arrow[d, "f_{k+1}"'] & X_k \arrow[d, "f_k"] \\ Y_{k+1} \arrow[r, "\rho_k"] & Y_k \end{tikzcd}$$

are commutative.

Proof. Suppose $s' \in T_{k+2}$ is a child of $s \in T_{k+1}$. Then $f_{k+1}(s') = t'$ for t' such that $f\,(Q_s) \subset P_{t'}$. Now $Q_{s'} \subset Q_s$ and $P_{t'} \subset P_t$, where t is the parent of t'. Since $f\,(Q_s) \subset P_{\tilde{t}}$ for a unique $\tilde{t} \in P_{k+1}$, the subsets P_r, $r \in P_{k+1}$ of X are disjoint, and $f\,(Q_s) \cap P_t \supset P_{t'} \neq \emptyset$ we get that $f\,(Q_s) \subset P_t$ and $f_k(s) = t$.

Now suppose $x \in Q_s$, where $s \in S_k$ is a leaf. Then $f_{k+1}(x) = f(x) \in P_t$, where $t = f_k(s)$. Therefore $f_k \rho_k(x) = f_k(s) = t = \rho_k\,(f(x)) = \rho_k f_{k+1}(x)$. $\square$

Suppose we have nested partition $\{\, Q_s \subset X \mid s \in S\}$, $\{\, P_t \subset Y \mid t \in T\}$ and maps $f_k\colon X_k \to Y_k$ such that the diagrams

$$\begin{tikzcd} X_{k+1} \arrow[r, "\rho_k"] \arrow[d, "f_{k+1}"'] & X_k \arrow[d, "f_k"] \\ Y_{k+1} \arrow[r, "\rho_k"] & Y_k \end{tikzcd}$$

are commutative. Define $f\colon X \to Y$ as follows. For every $x \in X$ there is a unique leaf $s \in S$ such that $x \in Q_s$. Suppose $s \in S_k$. Set $f(x) := f_k(x)$.

Proposition 2. *The map $f\colon X \to Y$ defined above respects partitions $\{\, Q_s \subset X \,|\, s \in S\}$ and $\{\, P_t \subset Y \,|\, t \in T\}$.*

Proof. Let $x \in Q_s \subset X$. There is a unique leaf s' of S such that $x \in Q_{s'}$. It is clear that s' is a descendant of s. Suppose $s' \in S_l$ and $s \in S_k$, $l \geq k$. Then $x \in X_l$, $s' \in X_{l-1}$ and $s \in X_{k-1}$. We consider the commutative diagram

$$\begin{tikzcd} X_l \arrow[r, "\rho_{l-1}"] \arrow[d, "f_l"'] & X_{l-1} \arrow[r, "\rho_{k-1}\circ\cdots\circ\rho_{l-2}"] \arrow[d, "f_{l-1}"'] & X_{k-1} \arrow[d, "f_{k-1}"] \\ Y_l \arrow[r, "\rho_{l-1}"] & Y_{l-1} \arrow[r, "\rho_{k-1}\circ\cdots\circ\rho_{l-2}"] & Y_{k-1}. \end{tikzcd}$$

From this diagram it follows that $f(x) = f_l(x) \in P_{f_{k-1}(s)}$. Since x was an arbitrary element of Q_s we get that $f\,(Q_s) \subset P_{f_{k-1}}(s)$. □

We say that a nested partition $\{\, P_t \,|\, t \in T\}$ of X is *uniformly nested* if all leaves of T have depth k for some $k \in \mathbb{N}$ and there are natural numbers n_1, n_2, $\ldots$, n_k such that $|P_t| = n_j$ for all $t \in T_j$. We will call $(n_1, \ldots, n_k)$ a *type* of the uniformly nested partition. We define the *standard* uniformly nested partition $I(\tilde{n})$ of type $\tilde{n} = (n_1, \ldots, n_k)$ as follows

$$I\left(\tilde{n}\right)_j := [1..n_1] \times \cdots \times [1..n_j]$$

for $0 \leq j \leq k$ and

$$\rho_j\colon I\left(\tilde{n}\right)_{j+1} \to I(\tilde{n})_j$$

to be the projection on the first j coordinates. It is clear that every uniformly nested partition of type $\tilde{n}$ is isomorphic to $I(\tilde{n})$. Since endomorphism monoids of isomorphic objects are isomorphic we will concentrate on the endomorphism monoid of $I\left(\tilde{n}\right)$. We will denote it by $\mathcal{P}\left(\tilde{n}\right)$. For every $v \in I\left(\tilde{n}\right)_{j-1}$ and $f \in \mathcal{P}\left(\tilde{n}\right)$ define $f\left[v\right] : [1..n_j] \to [1..n_j]$ by the requirement

$$f_j\left(v, i\right) = \left(f_{j-1}\left(v\right), f[v]\left(i\right)\right).$$

Proposition 3. *For every f, $g \in \mathcal{P}\left(\tilde{n}\right)$, $v \in I\left(\tilde{n}\right)_{j-1}$ we have $\left(fg\right)\left[v\right] = f\left[g_{j-1}\left(v\right)\right] \circ g[v]$.*

Proof. We have

$$\begin{aligned} \left(fg\right)_j\left(v, i\right) &= f_j\left(g_j\left(v, i\right)\right) \\ &= f_j\left(g_{j-1}\left(v\right), g[v]\left(i\right)\right) \\ &= \left(f_{j-1}\left(g_{j-1}\left(v\right)\right), f\left[g_{j-1}\left(v\right)\right]\left(g[v]\left(i\right)\right)\right) \\ &= \Big(\left(fg\right)_{j-1}\left(v\right), \left(f\left[g_{j-1}\left(v\right)\right] g\left[v\right]\right)\left(i\right)\Big). \end{aligned}$$

□

We can recover $f \in \mathcal{P}(\tilde{n})$ from the collection

$$\left\{ f[v] \colon [1..n_j] \to [1..n_j] \,\middle|\, v \in I(\tilde{n})_{j-1} \right\}.$$

In fact we can define $f_1 := f[*]$, where $*$ is the unique element of $I(\tilde{n})_0$. Then we proceed by induction and define $f_j(v,i) := (f_{j-1}(v), f[v](i))$.

For every $g \colon [1..n_j] \to [1..n_j]$ and $v \in I(\tilde{n})_{j-1}$ we define $[f,v] \in \mathcal{P}(\tilde{n})$ by

$$[g,v][w] := \begin{cases} f, & v = w \\ \mathrm{id}, & \text{otherwise.} \end{cases}$$

Note that for $v_1,\ v_2 \in I(\tilde{n})_{j-1}$ the elements $[g,v_1]$ and $[g,v_2]$ commute. We define $t_j(f)$ to be the product of elements $[f[v],v]$ where v ranges over $I(\tilde{n})_{j-1}$. As all this elements pairwise commute the order in such a product does not play any role. We have

$$t_j(f)[v] = \begin{cases} f[v], & v \in I(\tilde{n})_{j-1} \\ \mathrm{id}, & \text{otherwise.} \end{cases}$$

Proposition 4. *Let $v \in I(\tilde{n})_{j-1}$ and $g \colon [1..n_j] \to [1..n_j]$. Then for every $0 \le s \le j-1$ we have $[g,v]_s = \mathrm{id}$.*

Proof. Let ID be the identity map of $I(\tilde{n})$. Then $\mathrm{ID}_s = \mathrm{id}$ for every $0 \le s \le k$. Therefore for every $w \in I(\tilde{n})_s$ we have $\mathrm{ID}[w] = \mathrm{id} \colon [1..n_j] \to [1..n_j]$. Since the component f_s of $f \in \mathcal{P}(\tilde{n})$ can be recovered from the maps $f[w]$, $w \in I(\tilde{n})_r$, $r \le s$, and $[g,v][w] = \mathrm{id} = \mathrm{ID}[w]$ for all $w \in I(\tilde{n})_r$, $r \le j-1$, we see that $[g,v]_s = \mathrm{ID}_s = \mathrm{id}$ for all $s \le j-1$. □

Corollary 1. *For every $f \in \mathcal{P}(\tilde{n})$ and $1 \le s < j \le k$ we have $t_j(f)_s = \mathrm{id}$.*

Proposition 5. *Let $g_1,\ g_2$ be endomorphisms of $[1..n_j]$ and $v \in I)(\tilde{n})_{j-1}$. Then $[g_1,v][g_2,v] = [g_1g_2,v],$.*

Proof. Follows from the straightforward computation. □

Now we will prove

Proposition 6. *For every $f \in \mathcal{P}(\tilde{n})$ we have*

$$f = t_k(f) \circ \ldots t_1(f).$$

Proof. We write t_j instead of $t_j(f)$. We have

$$\begin{aligned}(t_k \circ \dots t_1)[v] &= t_k\left[\left(t_{k-1 \circ \dots \circ t_1}(v)\right)\right] \circ (t_{k-1} \circ \dots \circ t_1)[v] \\ &= (t_{k-1} \circ \dots \circ t_1)[v] \\ &= \dots \\ &= (t_j \circ \dots t_1)[v] \\ &= (t_j \circ \dots \circ t_2)\left[(t_1)_{j-1}(v)\right] \circ t_1[v] \\ &= (t_j \circ \dots \circ t_2)[v] \\ &= \dots \\ &= t_j[v] = f[v].\end{aligned}$$

□

2 Relative rank

Let S be a semigroup and $P\colon S \to \{\text{True}, \text{False}\}$ a predicate on S. We say that P is *primitive* if

$$\forall a, b \in S : P(ab) \Leftrightarrow P(a) \mathbin{\&} P(b).$$

Examples 1) Let R be a commutative ring and p a primitive ideal in R. Then the predicate $P(x) := (x \notin p)$ is a primitive predicate on the multiplicative semigroup of R. This example explains our terminology.
2) Let $\mathcal{C}$ be a category and X an object of $\mathcal{C}$. Then the predicate

$$f \text{ is an isomorphism}$$

is a primitive predicate on $\mathcal{C}(X, X)$.
3) If P_1 and P_2 are primitive predicates then $P_1 \& P_2$ is primitive.

Suppose P is a primitive predicate on S. We denote by S_P the subset of S of the elmements for which P is true. Then S_P is subsemigroup of S.

Proposition 7. *Let P be a primitive predicate on S. Then*

$$\operatorname{rk}(S) = \operatorname{rk}(S_P) + \operatorname{rk}(S : S_P).$$

Proof. It is obvious that $\operatorname{rk}(S) \le \operatorname{rk}(S_P) + \operatorname{rk}(S : S_P)$. Now let X be a generating set of X such that $|X| = \operatorname{rk}(S)$. Denote by X_P the subset $\{\, x \in X \mid P(x)\}$ of S_P. Then X_P generates S_P. In fact, let $s \in S_P$. Then $s = x_1 \dots x_m$ for some $x_i \in X$. Now

$$P(s) \Leftrightarrow P(x_1 \dots x_m) \Leftrightarrow P(x_1) \mathbin{\&} \cdots \mathbin{\&} P(x_m).$$

Thus $x_i \in X_P$ for every $1 \le i \le m$. This shows that s is an element of subsemigroup generated by X.

It is clear that $S_P \cup (X \setminus X_P)$ generates S, since already its subset $X = X_P \cup (X \setminus X_P)$ generates S. Therefore

$$\mathrm{rk}\,(S) = |X| = |X \setminus X_P| + |X_P| \geq \mathrm{rk}\,(S : S_P) + \mathrm{rk}\,(S_P)\,.$$

□

For every $1 \leq j \leq k$ we define the predicate P_j on $\mathcal{P}\,(\tilde{n})$ by

$$P_j\,(f) = \begin{cases} \text{True}, & f_j \text{ is invertible} \\ \text{False}, & \text{otherwise.} \end{cases}$$

These predicates are primitive. We shall denote $\mathcal{P}\,(\tilde{n})_{P_j}$ by $\mathcal{P}_j\,(\tilde{n})$.

Proposition 8. *We have $P_j\,(f) \Rightarrow P_{j-1}\,(f)$ and therefore $\mathcal{P}_j\,(\tilde{n}) \subset \mathcal{P}_{j-1}\,(\tilde{n})$.*

Proof. Since f_{j-1} is an endomorphism of finite set it is non-invertible if and only if there are two different elements v_1, v_2 of $I\,(\tilde{n})_{j-1}$ such that $f_{j-1}\,(v_1) = f_{j-1}\,(v_2) = v$. We consider restriction $\overline{f}_j$ of f_j on $\{v_1, v_2\} \times [1..n_j]$. The image of $\overline{f}_j$ is a subset of $\{v\} \times [1..n_j]$. Thus $\overline{f}_j$ is not injective and therefore $P_j\,(f) =$ False. □

Proposition 9. *Suppose $v \in I\,(\tilde{n})_{j-1}$ and $g\colon [1..n_j] \to [1..n_j]$. Then $[g,v]_j$ is invertible if and only if g is invertible. Moreover, if g is invertible then $[g,v] \in \mathcal{P}_k\,(\tilde{n})$.*

Then $[g,v] \in \mathcal{P}_k\,(\tilde{n})$ if and only if g is invertible.

Proof. For $s \leq j-1$ we know by Proposition 4 that $[g,v]_s = \mathrm{id}$. Now

$$[g,v]_j\,(w,i) = ([g,v]_{j-1}\,(w)\,, [g,v]\,[w]\,(i)) \begin{cases} (w,i)\,, & w \neq v \\ (w, g\,(i))\,, & w = v. \end{cases} \tag{1}$$

This shows that $[g,v]_j$ is invertible if and only if g is invertible.

Now we suppose that g is invertible. Then $[g,v]_j$ is invertible. Assume we showed that $[g,v]_r$ is invertible for all $j \leq r \leq s$. Then

$$[g,v]_{s+1}\,(w,i) = ([g,v]_s\,(w)\,, i)\,.$$

As $[g,v]_s$ is invertible by assumption, it follows that $[g,v]_{s+1}$ is invertible as well. □

Theorem 1. *For any $1 \leq j \leq k$ we have*

$$\mathrm{rk}\,(\mathcal{P}_{j-1}\,(\tilde{n}) : \mathcal{P}_j\,(\tilde{n})) = 1.$$

Proof. Since the inclusion of $\mathcal{P}_j\,(\tilde{n})$ in $\mathcal{P}_{j-1}\,(\tilde{n})$ is proper, the rank in question is at least 1. Define $\tau\colon [1..n_j] \to [1..n_j]$ by

$$\tau\,(i) = \begin{cases} 2, & i = 1 \\ i, & \text{otherwise.} \end{cases}$$

Denote $(1,\ldots,1)\in I(\tilde{n})_{j-1}$ by u. We claim that $[\tau,u]$ generates $\mathcal{P}_{j-1}(\tilde{n})$ over $\mathcal{P}_j(\tilde{n})$.

Let $f\in\mathcal{P}_j(\tilde{n})$. By Proposition 6 we have

$$f=t_k(f)\circ\cdots\circ t_1(f),$$

where

$$t_s(f)=\prod_{v\in I(\tilde{n})_{s-1}}[f[v],v].$$

Suppose $s\geq j+1$. Then by Corollary 1 $t_s(f)=\mathrm{id}$ and therefore $t_s(f)\in\mathcal{P}_j(\tilde{n})$.

Suppose $s\leq j-1$ and $v\in I(\tilde{n})_{s-1}$. Since $f\in\mathcal{P}_{j-1}(\tilde{n})$ if follows that $[f[v],v]\in\mathcal{P}_{j-1}(\tilde{n})$. By Proposition 8 $[f[v],v]\in\mathcal{P}_s(\tilde{n})$ and therefore $[f[v],v]_s$ is invertible. By Proposition 9 $f[v]$ is an automorphism of $[1..n_s]$. By the same proposition $[f[v],v]\in\mathcal{P}_k(\tilde{n})\subset\mathcal{P}_j(\tilde{n})$. Thus $t_s(f)\in\mathcal{P}_j(\tilde{n})$ for $s\leq j-1$.

Let $v\in I(\tilde{n})_{j-1}$. Then $f[v]:\ [1..n_j]\to[1..n_j]$ can be written as a product

$$g_1\tau g_2\tau\ldots\tau g_l$$

for some $l\in\mathbb{N}$ and automorphisms g_i of $[1..n_j]$. By Proposition 5 we have

$$[f[v],v]=[g_1,v][\tau,v]\ldots[\tau,v][g_l,v].$$

By Proposition 9 $[g_i,v]\in\mathcal{P}_j(\tilde{n})$ for $1\leq i\leq l$. Thus it is enough to show that $[\tau,v]$ belongs to a semigroup of $\mathcal{P}_{j-1}(\tilde{n})$ generated by $\mathcal{P}_j(\tilde{n})$ and $[\tau,u]$. Suppose $v=(v_1,\ldots,v_j)$, $1\leq v_i\leq n_i$. Define $h\in\mathcal{P}(\tilde{n})$ by

$$h_s(w):=\begin{cases}(v_1,\ldots,v_s) & w=(1,\ldots,1)\\ (1,\ldots,1) & w=(v_1,\ldots,v_s)\text{ if }s\leq j-1\\ w & \text{otherwise}\end{cases}$$

$$h_s(w):=\begin{cases}(v_1,\ldots,v_{j-1},w_j,\ldots,w_s) & (w_1,\ldots,w_{j-1})=u\\ (1,\ldots,1,w_j,\ldots,w_s) & (w_1,\ldots,w_{j-1})=v\text{ if }s\geq j.\\ w & \text{otherwise}\end{cases}$$

Then $h^2=\mathrm{ID}$ and thus $h\in\mathcal{P}_j(\tilde{n})$. Moreover $[\tau,v]=h[\tau,u]h$. This finishes the proof. ∎

Corollary 2. *We have*

$$\mathrm{rk}(\mathcal{P}(\tilde{n}))=k+\mathrm{rk}(\mathcal{P}(\tilde{n}):\mathcal{P}_k(\tilde{n})).$$

Proof. Apply Theorem 1 and Proposition 7. ∎

3 Generators for wreath product

We will multiply permutations from left to right, thus

$$(1,2)\,(2,3) = (1,3,2)\,.$$

Correspondingly, if $\pi \in S_m$ and $i \in \{1,\dots,m\}$, then the result of application of π to i will be denoted by $i\pi$.

Let G be a group. We define a left action of S_m on G^m by

$$\pi\,(g_1, g_2, \dots, g_m) = (g_{1\pi}, g_{2\pi}, \dots, g_{m\pi})\,.$$

Then the multiplication in the wreath product $G \wr S_m = G^m \rtimes S_m$ is given by

$$(h_1, \dots, h_m)\,\pi\,(g_1, g_2, \dots, g_m)\,\sigma = (h_1 g_{1\pi}, h_2 g_{2\pi}, \dots, h_m g_{m\pi})\,\pi\sigma.$$

We will consider G^m and S_m as subgroups of $G \wr S_m$. Denote by $[-,i]$ the embedding of G into the j-th component of G^m.Then the set

$$\{\,[g,i]\,|\,g \in G,\ i \in \{1,\dots,m\}\} \cup S_m$$

generates $G \wr S_m$. In the following we will use that for a different i and j the elements $[g,i]$ and $[h,j]$ of $G \wr S_m$ commute, and that for $\pi \in S_m$

$$\pi[g,i] = [g, i\pi^{-1}]\pi.$$

Proposition 10. *Suppose $\{g_1,\dots,g_k\}$ and $\{\pi_1,\dots,\pi_l\}$ are generating sets of G and S_m respectively. Then for any multi-index $(i_1,\dots,i_k)$, $1 \le i_t \le m$, the set*

$$X = \{\,[g_t, i_t]\,|\,1 \le t \le k\} \cup \{\pi_1,\dots,\pi_l\}$$

generates $G^m \wr S_m$.

Proof. Denote by H the subgroup of G generated by X. Then $S_m \subset H$ as $\{\pi_1,\dots,\pi_l\} \subset X$ and $\{\pi_1,\dots,\pi_l\}$ generates S_m. Now for every $1 \le t \le k$ and $1 \le j \le m$ we have

$$(i_t, j)[g_t, i_t](i_t, j) = [g_t, j] \in H.$$

Since the $\{\,[g_t, j]\,|\,1 \le t \le k\}$ generates subgroup $[G,j]$ of $G \wr S_m$ we get that

$$\{\,[g,i]\,|\,g \in G,\ i \in \{1,\dots,m\}\} \cup S_m \subset H$$

and therefore $H = G \wr S_m$. □

Now we prove two lemmas that show how the elements of the form $[g,i]$ and π can be recovered from the elements of the form $[g,i]\pi$.

Lemma 1. *Suppose $g \in G$ and $\pi \in S_m$ have coprime orders. Let $i \in \{1,\dots,m\}$ be such that $i\pi = i$. Then $[g,i]$ and π are elements of the cyclic subgroup generated by $[g,i]\pi$.*

Proof. Note that $[g, i]$ and π commute since $\pi[g, i] = [g, i\pi]\pi = [g, i]\pi$. Let k and l be the orders of g and π respectively. Since k and l are coprime there are p and q such that $pk + ql = 1$. Now $([g, i]\pi)^{pk} = \left[g^{pk}, i\right]\pi^{1-ql} = \pi$ and $([g, i]\pi)^{ql} = \left[g^{1-pk}, i\right]\pi^{ql} = [g, i]$. $\square$

Lemma 2. *Let g be an element of G of odd order and $\sigma \in G$ of order 2. Denote by H the subgroup of $G \wr S_m$ generated by*

$$\begin{aligned} a =&[\sigma, 2]\,(1, \ldots, m) \\ b =&[g, 3]\,(1, 2)\,. \end{aligned}$$

Then $[g, 3]$, $[\sigma, 1]$, $(1, \ldots, m)$ and $(1, 2)$ are elements of H.

Proof. Since the order of g is odd, the order of $(1, 2)$ is 2, and $[g, 3]\,(1, 2) \in H$ it follows from Lemma 1 that $[g, 3]$ and $(1, 2)$ are elements of H.

Now we have that

$$\begin{aligned} a^m &= ([\sigma, 2]\,(1, \ldots, m))^m = [\sigma, 2][\sigma, 3] \ldots [\sigma, m][\sigma, 1]\,(1, \ldots, m)^m \\ &= (\sigma, \sigma, \ldots, \sigma) \in H. \end{aligned}$$

Consider the product

$$\begin{aligned} ab &= [\sigma, 2]\,(1, \ldots, m)\,[g, 3]\,(1, 2) = [\sigma, 2][g, 3\,(1, \ldots, m)^{-1}]\,(2, \ldots, m) \\ &= [\sigma g, 2]\,(2, \ldots, m)\,. \end{aligned}$$

Therefore

$$\begin{aligned} (ab)^{m-1} &= ([\sigma g, 2]\,(2, \ldots, m))^{m-1} = [\sigma g, 2][\sigma g, 3] \ldots [\sigma g, m]\,(2, \ldots, m)^{m-1} \\ &= (e, \sigma g, \ldots, \sigma g) \in H \end{aligned}$$

and, using $\sigma^2 = e$,

$$a^m\,(ab)^{m-1} = (\sigma, g, \ldots, g) \in H.$$

Let l be the order of g. Then since l is odd and the order of σ is 2 we have $\sigma^l = \sigma$. Thus

$$\left(a^m\,(ab)^{m-1}\right) = (\sigma, e, \ldots, e) = [\sigma, 1] \in H.$$

Note that $a^{-1} = [\sigma, 3]\,(1, m, \ldots, 2)$. Therefore

$$\begin{aligned} a^{-1}[\sigma, 1]a^2 &= [\sigma, 3]\,(1, m, \ldots, 2)\,[\sigma, 1][\sigma, 2]\,(1, \ldots, m)\,[\sigma, 1]\,(1, \ldots, m) \\ &= [\sigma, 3][\sigma, 2][\sigma, 3]\,(1, m, \ldots, 2)\,(1, 2, \ldots, m)\,[\sigma, 2]\,(1, \ldots, m) \\ &= [\sigma, 2][\sigma, 2]\,(1, 2, \ldots, m) = (1, \ldots, m) \in H. \end{aligned}$$

$\square$

4 Iterated wreath product

In this section we identify $\mathcal{P}_k(\tilde{n})$ with an iterated wreath product and show that its rank is k.

Let $\mathbb{Z}_2 = e, \sigma$ be the cyclic group of order 2. Denote by ε_j the parity homomorphism from S_{n_j} to $\mathbb{Z}_2$. Define $\varepsilon\colon \mathcal{P}_k(\tilde{n}) \to \mathbb{Z}_2^k$ by

$$\varepsilon(f) = (\varepsilon_1(f_1), \ldots, \varepsilon_k(f_k)).$$

Proposition 11. *The homomorphism ε is surjective and therefore* $\operatorname{rk}(\mathcal{P}_k(\tilde{n})) \geq k$.

Proof. Let $g_j = [(1,2), u_j]$, where $u_j = (1, \ldots, 1) \in I(\tilde{n})_{j-1}$. Then by Proposition 4 for $s \leq j-1$

$$\varepsilon_s((g_j)_s) = \varepsilon_s(\mathrm{id}) = e.$$

Now by (1)

$$(g_j)_j(w,i) = \begin{cases} (w,i) & w \neq u_j \\ (w,(1,2)(i)) & w = u_j. \end{cases}$$

Thus $(g_j)_j$ swaps two elements $(u_j,1)$ and $(u_j,2)$ of $I(\tilde{n})_j$. Therefore $\varepsilon_j\left((g_j)_j\right) = \sigma$. Since the elements

$$\begin{aligned} (\sigma, *, \ldots, *) \\ (e, \sigma, *, \ldots, *) \\ \ldots \\ (e, \ldots, e, \sigma) \end{aligned}$$

generate $\mathbb{Z}_2^k$ by Gauss elimination process, we see that ε is surjective. $\square$

Let $\overline{n} = (n_2, \ldots, n_k)$. We can identify $\mathcal{P}_k(\tilde{n})$ with $\mathcal{P}_{k-1}(\overline{n}) \wr S_{n_1}$ as follows. Let $f \in \mathcal{P}_k(\tilde{n})$ and $1 \leq i \leq n_1$. Define $f(i) \in \mathcal{P}_{k-1}(\overline{n})$ from the equalities

$$\left(f(i)_j(v), f_1(i)\right) = f_{j+1}(v,i).$$

Then

$$f \mapsto (f(1), \ldots, f(n_1))\, f_1$$

is an isomorphism from $\mathcal{P}_k(\tilde{n})$ to $\mathcal{P}_{k-1}(\overline{n}) \wr S_{n_1}$. By iteration we get that

$$\mathcal{P}_k(\tilde{n}) \cong S_{n_k} \wr S_{n_{k-1}} \wr \cdots \wr S_{n_1}.$$

Let $\sigma \in S_{n_j}$ and $(v_1, \ldots, v_{j_1}) \in I(\tilde{n})_{j-1}$. Then upon this identification

$$[\sigma, u] = [[[\sigma, v_{j-1}], \ldots], v_1].$$

For every j we define an element τ_j of S_{n_j} by

$$\tau_j = \begin{cases} (1, \ldots, n_j) & n_j \text{ is odd} \\ (2, \ldots, n_j) & n_j \text{ is even.} \end{cases}$$

Note that the order of τ_j is odd for every j and that the elements $(1,2)$, τ_j generate S_{n_j}.

Define

$$g_j = [\tau_{j+1}, 3]\,(1,2) \in S_{n_k} \wr \cdots \wr S_{n_j},$$

and

$$\tilde{g}_j = [[g_j, 3], \ldots, 3] \in S_{n_k} \wr \cdots \wr S_{n_1},$$

Define

$$\tilde{g} = [[[(1,2), 2], \ldots], 2]\, S_{n_k} \wr \cdots \wr S_{n_1}.$$

Theorem 2. *The set* $X = \{\tilde{g}_1, \ldots, \tilde{g}_{k-1}, \tilde{g}\}$ *generates* $G = S_{n_k} \wr \cdots \wr S_{n_1}$ *and therefore* $\operatorname{rk}(G) \leq k$.

Proof. Let H be a subgroup of G generated by X. Denote by u_j the element $(3, \ldots, 3)$ of $I(\tilde{n})_{j-1}$.

As $\tilde{g}_j \in H$, it follows from Lemma 1 that for $1 \leq j \leq k-1$

$$[(1,2), u_j] \in H$$

and for $2 \leq j \leq k$

$$[\tau_j, u_j] \in H.$$

Now we apply Lemma 2 to $\tilde{g}_1$ and $\tilde{g}$. We get that $(1, \ldots, n_1)$ is an element of H. By iteration of Proposition 10 we get that these elements generate the whole group G. □

Theorem 3. *We have* $\operatorname{rk}(\mathcal{P}(\tilde{n})) = 2k$.

Proof. From Theorem 2 and Proposition 11 it follows that $\operatorname{rk}(\mathcal{P}_k(\tilde{n})) = k$. Now apply Corollary 2. □

References

[1] João Araújo and Csaba Schneider, *The rank of the endomorphism monoid of a uniform partition*, Semigroup Forum **78** (2009), no. 3, 498–510. MR MR2511780 (2010d:20069)